

QUANTUM GENERALIZED CLUSTER ALGEBRAS AND QUANTUM DILOGARITHMS OF HIGHER DEGREES

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ABSTRACT. We extend the notion of the quantization of the coefficients of the ordinary cluster algebras to the generalized cluster algebras by Chekhov and Shapiro. In parallel to the ordinary case, it is tightly integrated with certain generalizations of the ordinary quantum dilogarithm, which we call the quantum dilogarithms of higher degrees. As an application, we derive the identities of these generalized quantum dilogarithms associated with any period of quantum Y -seeds.

1. INTRODUCTION

The *generalized cluster algebras* were introduced by Chekhov and Shapiro [CS14]. They naturally generalize the (ordinary) cluster algebras by Fomin and Zelevinsky [FZ03]. The main feature of the generalized cluster algebras is the appearance of *polynomials* in the exchange relations of cluster variables and coefficients, instead of *binomials* in the ordinary case. Generalized cluster algebras naturally appear so far in Poisson dynamics [GSV03], Teichmüller theory [CS14], representation theory [Gle14], exact WKB analysis [IN14], etc. It has been shown in [CS14, Nak14] that essentially all important properties of the ordinary cluster algebras are naturally extended to the generalized ones.

In this note we demonstrate that the notion of *quantum cluster algebras* is also extended to the generalized ones. To be more precise, there are two kinds of formulations of quantum cluster algebras, the one quantizing the *cluster variables* by [BZ05] and the one quantizing the *coefficients* by [FG09a, FG09b], and it is known that they are closely related to each other. Here, we concentrate on the latter one. As shown by [FG09a, FG09b], in the ordinary case, the quantization of the coefficients is tightly integrated with the *quantum dilogarithm* [FV93, FK94]. Similarly, in the generalized case, it is tightly integrated with certain generalizations of the quantum dilogarithm, which we call the *quantum dilogarithms of higher degrees*. As an application, we derive the identities of these generalized quantum dilogarithms associated with any period of quantum Y -seeds, which are also parallel to the ones in the ordinary case.

The main message of this note is that the fundamental (and perhaps all) features of the quantum cluster algebras are also extended to the generalized ones.

2. QUANTUM DILOGARITHMS OF HIGHER DEGREES

To begin with, let us recall some basic facts about the dilogarithm, the q -dilogarithm, and the quantum dilogarithm. The *dilogarithm* $\text{Li}_2(x)$ is defined by

$$(2.1) \quad \text{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}.$$

Let q be a formal variable. The q -*dilogarithm* is defined as follows.

$$(2.2) \quad \mathcal{L}_{2,q}(x) = \sum_{n=1}^{\infty} \frac{x^n}{n(q^n - q^{-n})} = \frac{1}{q - q^{-1}} \sum_{n=1}^{\infty} \frac{x^n}{n[n]_q},$$

where $[n]_q = (q^n - q^{-n})/(q - q^{-1})$ is the standard q -number. The power series (2.2) converges for $|x| < 1$ and $|q| < 1$, and the following asymptotic behavior holds when $q \rightarrow 1^-$,

$$(2.3) \quad \mathcal{L}_{2,q}(x) \sim \frac{\text{Li}_2(x)}{q^2 - 1} \sim \frac{\text{Li}_2(x)}{\log q^2}.$$

This is clear from the second expression of $\mathcal{L}_{2,q}(x)$ in (2.2) and the property $\lim_{q \rightarrow 1} [n]_q = n$.

Following [FV93, FK94] (up to some convention), we introduce the *quantum dilogarithm* $\Psi_q(x)$, which is a formal power series in x with coefficients in $\mathbb{C}(q)$, as follows.

$$(2.4) \quad \Psi_q(x) = \prod_{m=0}^{\infty} (1 + q^{2m+1}x)^{-1}.$$

In particular, the quantum dilogarithm $\Psi_q(x)$ should be distinguished from the q -dilogarithm $\mathcal{L}_{2,q}(x)$. The formal power series $\Psi_q(x)$ is characterized by the following recursion relation with initial condition,

$$(2.5) \quad \Psi_q(0) = 1, \quad \Psi_q(q^{\pm 2}x) = (1 + qx)^{\pm 1} \Psi_q(x),$$

where two relations in the latter equality are equivalent to each other. A little confusingly, the quantum dilogarithm is actually the exponential of the q -dilogarithm; namely,

$$(2.6) \quad \Psi_q(x) = \exp(-\mathcal{L}_{2,q}(-x)).$$

This is easily shown by using the recursion relation (2.5).

Alternatively, one may define the *dilogarithm* $\text{Li}_2(x)$ by the integral

$$(2.7) \quad \text{Li}_2(x) = -\int_0^x \log(1-y) \frac{dy}{y} = -\int_0^{-x} \log(1+y) \frac{dy}{y}.$$

Then we have

$$(2.8) \quad \begin{aligned} \log \Psi_q(x) &= -\sum_{m=0}^{\infty} \log(1 + q^{2m+1}x) \\ &= \frac{-1}{1-q^2} \sum_{m=0}^{\infty} \log(1 + q^{2m+1}x) \frac{q^{2m+1}x - q^{2m+3}x}{q^{2m+1}x} \\ &\sim \frac{-1}{1-q^2} \int_0^x \log(1+y) \frac{dy}{y} = \frac{1}{1-q^2} \text{Li}_2(-x) \quad (q \rightarrow 1^-). \end{aligned}$$

This completely agrees with (2.3) and (2.6).

Now let us generalize the quantum dilogarithm $\Psi_q(x)$ to the ones with higher degrees. For any field F , let $F(q)$ be the field of the rational functions in the variable q .

Definition 2.1. Let F be a field, let d be a positive integer, and let $\mathbf{z} = (z_1, \dots, z_{d-1})$ be a $d-1$ -tuple of elements in F . When $d = 1$, \mathbf{z} is regarded as the empty sequence $()$. We set $z_0 = z_d = 1$. Then, we define a formal power series $\Psi_{d,\mathbf{z},q}(x)$ in x with coefficients in $F(q)$ as follows:

$$(2.9) \quad \Psi_{d,\mathbf{z},q}(x) = \prod_{m=0}^{\infty} \left(\sum_{s=0}^d z_s q^{s(2m+1)} x^s \right)^{-1}.$$

When $d = 1$, it is the usual quantum dilogarithm $\Psi_{1,(),q}(x) = \Psi_q(x)$. We call $\Psi_{d,\mathbf{z},q}(x)$ the *quantum dilogarithm of degree d with coefficients \mathbf{z}* .

Proposition 2.2. *The formal power series $\Psi_{d,\mathbf{z},q}(x)$ is characterized by the following recursion relation with initial condition:*

$$(2.10) \quad \Psi_{d,\mathbf{z},q}(0) = 1,$$

$$(2.11) \quad \Psi_{d,\mathbf{z},q}(q^{\pm 2}x) = \left(\sum_{s=0}^d z_s q^{\pm s} x^s \right)^{\pm 1} \Psi_{d,\mathbf{z},q}(x),$$

where two relations in (2.11) are equivalent to each other.

Proof. For example, we have

$$(2.12) \quad \begin{aligned} \Psi_{d,\mathbf{z},q}(q^2x) &= \prod_{m=0}^{\infty} \left(\sum_{s=0}^d z_s q^{s(2m+1)} q^{2s} x^s \right)^{-1} \\ &= \prod_{m=1}^{\infty} \left(\sum_{s=0}^d z_s q^{s(2m+1)} x^s \right)^{-1} = \left(\sum_{s=0}^d z_s q^s x^s \right)^{-1} \Psi_{d,\mathbf{z},q}(x). \end{aligned}$$

The rest of the properties are easily shown. \square

For any integer a , let us introduce the *sign function*

$$(2.13) \quad \text{sgn}(a) = \begin{cases} + & a > 0 \\ 0 & a = 0 \\ - & a < 0. \end{cases}$$

Here and below, we identify the signs \pm with numbers ± 1 .

The following formula will be useful later.

Proposition 2.3. *For any integer a , the following equality holds.*

$$(2.14) \quad \Psi_{d,\mathbf{z},q}(q^{2a}x) = \left(\prod_{m=1}^{|a|} \left(\sum_{s=0}^d z_s q^{\text{sgn}(a)(2m-1)s} x^s \right)^{\text{sgn}(a)} \right) \Psi_{d,\mathbf{z},q}(x).$$

Proof. This is obtained from (2.11) by induction on a . \square

In some cases the quantum dilogarithms of higher degrees are factorized by the ordinary quantum dilogarithm.

Proposition 2.4. *Factorization formula. Suppose that the following factorization*

$$(2.15) \quad \sum_{s=0}^d z_s x^s = \prod_{s=1}^d (1 - w_s x)$$

occurs for some $w_1, \dots, w_d \in F$. Then, we have

$$(2.16) \quad \Psi_{d, \mathbf{z}, q}(x) = \prod_{s=1}^d \Psi_q(-w_s x).$$

Proof. One can directly observe the factorization in (2.16) as

$$(2.17) \quad \Psi_{d, \mathbf{z}, q}(x) = \prod_{m=0}^{\infty} \left(\sum_{s=0}^d z_s q^{s(2m+1)} x^s \right)^{-1} = \prod_{m=0}^{\infty} \prod_{s=1}^d (1 - w_s q^{2m+1} x)^{-1}.$$

Alternatively, under the assumption (2.15), the right hand side of (2.16) satisfies (2.10) and (2.11). Thus, thanks to Proposition 2.3, we have (2.16). \square

Example 2.5. Let us consider the special case where $F = \mathbb{C}$ and the coefficients \mathbf{z} is trivial, i.e., $\mathbf{z} = \mathbf{1} := (1, \dots, 1)$. In this case we have the factorization

$$(2.18) \quad \sum_{s=0}^d x^s = \prod_{s=1}^d (1 - \omega^s x),$$

where

$$(2.19) \quad \omega = \exp(2\pi i / (d+1)).$$

Thus, by Proposition 2.4, we have

$$(2.20) \quad \Psi_{d, \mathbf{1}, q}(x) = \prod_{s=1}^d \Psi_q(-\omega^s x).$$

On the other hand, there is another factorization formula,

$$(2.21) \quad \Psi_{d, \mathbf{1}, q}(x) = \Psi_{q^{d+1}}(-x^{d+1}) \Psi_q(-x)^{-1}.$$

This is due to the following alternative expression of $\Psi_{d, \mathbf{1}, q}(x)$,

$$(2.22) \quad \Psi_{d, \mathbf{1}, q}(x) = \prod_{m=0}^{\infty} \frac{1 - q^{2m+1} x}{1 - (q^{2m+1} x)^{d+1}}.$$

Therefore, by (2.3) and (2.6), we have the following asymptotic behavior in the limit $q \rightarrow 1^-$:

$$(2.23) \quad \log \Psi_{d, \mathbf{1}, q}(x) \sim \frac{1}{1 - q^2} \sum_{s=1}^d \text{Li}_2(\omega^s x)$$

$$(2.24) \quad \sim \frac{1}{1 - q^2} \left(\frac{1}{d+1} \text{Li}_2(x^{d+1}) - \text{Li}_2(x) \right).$$

In fact, these two expressions coincide due to the well-known identity for $\text{Li}_2(x)$ called the *factorization formula* [Lew81, Eq. (1.14)],

$$(2.25) \quad \frac{1}{d+1} \text{Li}_2(x^{d+1}) = \sum_{s=0}^d \text{Li}_2(\omega^s x).$$

As a side remark, in view of (2.6), the expressions (2.20) and (2.21) imply the equality

$$(2.26) \quad \mathcal{L}_{2,q^{d+1}}(x^{d+1}) = \sum_{s=0}^d \mathcal{L}_{2,q}(\omega^s x),$$

which is regarded as the q -analogue of (2.25). The equality (2.26) is also obtained directly from (2.2) and the equality

$$(2.27) \quad \sum_{s=0}^d \omega^{sn} = \begin{cases} d+1 & n \equiv 0 \pmod{d+1} \\ 0 & n \not\equiv 0 \pmod{d+1}. \end{cases}$$

Example 2.6. Let us consider the case where $F = \mathbb{C}$ with arbitrary coefficients \mathbf{z} . Let us introduce the *dilogarithm of degree d with coefficients \mathbf{z}* by the integral,

$$(2.28) \quad \text{Li}_{2;d,\mathbf{z}}(x) = - \int_0^{-x} \log \left(\sum_{s=0}^d z_s y^s \right) \frac{dy}{y}.$$

Then, by the same calculation as in (2.8), we have the following asymptotic behavior,

$$(2.29) \quad \begin{aligned} \log \Psi_{d,\mathbf{z},q}(x) &= \frac{-1}{1-q^2} \sum_{m=0}^{\infty} \log \left(\sum_{s=0}^d z_s (q^{2m+1}x)^s \right) \frac{q^{2m+1}x - q^{2m+3}x}{q^{2m+1}x} \\ &\sim \frac{1}{1-q^2} \text{Li}_{2;d,\mathbf{z}}(-x) \quad (q \rightarrow 1^-). \end{aligned}$$

3. GENERALIZED MUTATIONS OF QUANTUM Y -SEEDS

In this section, following the idea of [FG09a, FG09b], we introduce the quantum version of the generalized mutation of generalized cluster algebras. Here, we use the formulation of generalized cluster algebras by [Nak14].

Let $B = (b_{ij})_{i,j=1}^n$ be a skew-symmetrizable integer matrix. Let $\mathbf{d} = (d_1, \dots, d_n)$ be an n -tuple of positive integers. For given B and \mathbf{d} , we arbitrarily choose an n -tuple of positive integers $\mathbf{r} = (r_1, \dots, r_n)$ such that

$$(3.1) \quad r_i d_i b_{ij} = -r_j d_j b_{ji}.$$

Such an \mathbf{r} exists (not uniquely) due to the skew-symmetrizable property of the matrix B . Let q continue to be a formal variable, and let $Y = (Y_i)_{i=1}^n$ be an n -tuple of noncommutative formal variables with commutation relation

$$(3.2) \quad Y_i Y_j = q^{2r_j d_j b_{ji}} Y_j Y_i.$$

The relation (3.2) makes sense due to the skew-symmetric property in (3.1). We call such a pair (B, Y) a *quantum Y -seed*.

We use the notation

$$(3.3) \quad q_i := q^{r_i d_i}, \quad i = 1, \dots, n.$$

Then, (3.2) is also written as

$$(3.4) \quad Y_i Y_j = q_j^{2b_{ji}} Y_j Y_i = q_i^{-2b_{ij}} Y_j Y_i.$$

Let F be any field. For the above $\mathbf{d} = (d_1, \dots, d_n)$ we arbitrarily choose a collection of elements in F ,

$$(3.5) \quad \mathbf{z} = (z_{i,s})_{i=1,\dots,n; s=1,\dots,d_i-1}$$

satisfying the *reciprocity condition* in [Nak14]

$$(3.6) \quad z_{i,s} = z_{i,d_i-s}.$$

(The use of symbol \mathbf{z} here slightly conflicts with the one in Definition 2.1, but we find that it is convenient.) Let us set $z_{i,0} = z_{i,d_i} = 1$. We also introduce the notation

$$(3.7) \quad \mathbf{z}_i = (z_{i,s})_{s=1,\dots,d_i-1}, \quad i = 1, \dots, n.$$

Under these notations we have the associated quantum dilogarithm $\Psi_{d_i, \mathbf{z}_i, q_i}(x)$ of degree d_i for each $i = 1, \dots, n$.

Below we assume that any element in F commutes with variables Y_i .

Definition 3.1. For a quantum Y -seed (B, Y) , the (\mathbf{d}, \mathbf{z}) -mutation (generalized mutation) $(B', Y') = \mu_k(B, Y)$ of (B, Y) at k is defined by

$$(3.8) \quad b'_{ij} = \begin{cases} -b_{ij} & i = k \text{ or } j = k \\ b_{ij} + d_k([-b_{ik}]_+ b_{kj} + b_{ik}[b_{kj}]_+) & i, j \neq k, \end{cases}$$

$$(3.9) \quad Y'_i = \begin{cases} Y_k^{-1} & i = k \\ q_i^{b_{ik}d_k[\varepsilon b_{ki}]_+} Y_i Y_k^{d_k[\varepsilon b_{ki}]_+} \times \prod_{m=1}^{|b_{ki}|} \left(\sum_{s=0}^{d_k} z_{k,s} q_k^{-\varepsilon \operatorname{sgn}(b_{ki})(2m-1)s} Y_k^{\varepsilon s} \right)^{-\operatorname{sgn}(b_{ki})} & i \neq k, \end{cases}$$

where $\varepsilon = \pm$, and

$$(3.10) \quad [a]_+ = \begin{cases} a & a > 0 \\ 0 & a \leq 0. \end{cases}$$

Actually, the right hand side of (3.9) does not depend on the choice of the sign ε (see Lemma 3.2 (i)). We call \mathbf{d} and \mathbf{z} the *mutation degrees* and the *frozen coefficients*, respectively, in accordance with [Nak14].

When we formally set $q = 1$, the relation (3.9) reduces to

$$(3.11) \quad Y'_i = \begin{cases} Y_k^{-1} & i = k \\ Y_i Y_k^{d_k[\varepsilon b_{ki}]_+} \left(\sum_{s=0}^{d_k} z_{k,s} Y_k^{\varepsilon s} \right)^{-b_{ki}} & i \neq k, \end{cases}$$

which is the generalized mutation of coefficients (y -variables) in generalized cluster algebras formulated in [Nak14]. On the other hand, when we set $d_k = 1$, it reduces to the ordinary mutation of quantum Y -seeds by Fock and Goncharov [FG09a, FG09b].

The following properties are easily checked.

Lemma 3.2. (i) The right hand side of (3.9) does not depend on the choice of the sign ε .

(ii) For the matrix B' , the condition

$$(3.12) \quad r_i d_i b'_{ij} = -r_j d_j b'_{ji}.$$

holds.

(iii) The (\mathbf{d}, \mathbf{z}) -mutation is involutive, i.e., $\mu_k(\mu_k(B, Y)) = (B, Y)$.

In the rest of the section, we will justify the relation (3.9) as a “good” quantization of the classical one (3.11) in the sense of [FG09a, FG09b].

To start, let us consider

$$(3.13) \quad \text{Ad}(\Psi_{d_k, \mathbf{z}_k, q_k}(Y_k^\varepsilon))^\varepsilon(Y_i) := \Psi_{d_k, \mathbf{z}_k, q_k}(Y_k^\varepsilon)^\varepsilon Y_i \Psi_{d_k, \mathbf{z}_k, q_k}(Y_k^\varepsilon)^{-\varepsilon},$$

which we call the *adjoint action* of $\Psi_{d_k, \mathbf{z}_k, q_k}(Y_k^\varepsilon)$ on quantum Y -variables.

The following is the key formula which connects the generalized mutation of quantum Y -seeds and the quantum dilogarithms of higher degrees.

Lemma 3.3.

$$(3.14) \quad \text{Ad}(\Psi_{d_k, \mathbf{z}_k, q_k}(Y_k^\varepsilon))^\varepsilon(Y_i) = Y_i \prod_{m=1}^{|b_{ki}|} \left(\sum_{s=0}^{d_k} z_{k,s} q_k^{-\varepsilon \text{sgn}(b_{ki})(2m-1)s} Y_k^{\varepsilon s} \right)^{-\text{sgn}(b_{ki})}.$$

Proof. For example, in the case $\varepsilon = +$,

$$(3.15) \quad \begin{aligned} \Psi_{d_k, \mathbf{z}_k, q_k}(Y_k) Y_i \Psi_{d_k, \mathbf{z}_k, q_k}(Y_k)^{-1} &= Y_i \Psi_{d_k, \mathbf{z}_k, q_k}(q_k^{-2b_{ki}} Y_k) \Psi_{d_k, \mathbf{z}_k, q_k}(Y_k)^{-1} \\ &= Y_i \prod_{m=1}^{|b_{ki}|} \left(\sum_{s=0}^{d_k} z_{k,s} q_k^{-\text{sgn}(b_{ki})(2m-1)s} Y_k^s \right)^{-\text{sgn}(b_{ki})}, \end{aligned}$$

where we used (3.2) and Proposition 2.3 in the first and second equalities, respectively. The case $\varepsilon = -$ can be shown in the same way. \square

The right hand side of (3.14), excluding the factor Y_i , is a part of (3.9), and for $d_k = 1$ it is called the “automorphism part” of (3.9) in [FG09a, FG09b].

Next let us consider the “monomial part” of (3.9). Let us set

$$(3.16) \quad Z_i^{(\varepsilon)} := \begin{cases} Y_k^{-1} & i = k \\ q_i^{b_{ik} d_k [\varepsilon b_{ki}]_+} Y_i Y_k^{d_k [\varepsilon b_{ki}]_+} & i \neq k. \end{cases}$$

By Lemma 3.3, the (\mathbf{d}, \mathbf{z}) -mutation (3.9) is expressed as the composition

$$(3.17) \quad Y'_i = \text{Ad}(\Psi_{d_k, \mathbf{z}_k, q_k}(Y_k^\varepsilon))^\varepsilon(Z_i^{(\varepsilon)}).$$

Lemma 3.4. *The following commutation relation holds.*

$$(3.18) \quad Z_i^{(\varepsilon)} Z_j^{(\varepsilon)} = q^{2r_j d_j b'_{ji}} Z_j^{(\varepsilon)} Z_i^{(\varepsilon)},$$

where b'_{ij} is given by (3.8).

Proof. This is easily verified by the case check. \square

Proposition 3.5. *Under the (\mathbf{d}, \mathbf{z}) -mutation in (3.9), the following commutation relation holds:*

$$(3.19) \quad Y'_i Y'_j = q^{2r_j d_j b'_{ji}} Y'_j Y'_i.$$

Proof. By Lemma 3.4 and (3.17), we have

$$(3.20) \quad \begin{aligned} Y'_i Y'_j &= \text{Ad}(\Psi_{d_k, \mathbf{z}_k, q_k}(Y_k^\varepsilon))^\varepsilon(Z_i^{(\varepsilon)} Z_j^{(\varepsilon)}) \\ &= \text{Ad}(\Psi_{d_k, \mathbf{z}_k, q_k}(Y_k^\varepsilon))^\varepsilon(q^{2r_j d_j b'_{ji}} Z_j^{(\varepsilon)} Z_i^{(\varepsilon)}) \\ &= q^{2r_j d_j b'_{ji}} Y'_j Y'_i. \end{aligned}$$

\square

Lemmas 3.3, 3.4, and Proposition 3.5 naturally extend the fundamental properties of the mutation of quantum Y -seeds in [FG09a, FG09b].

4. QUANTUM DILOGARITHM IDENTITIES OF HIGHER DEGREES

Let us give an application of generalized mutations of quantum Y -seeds to quantum dilogarithm identities of higher degrees. Since they are parallel to the one for ordinary quantum dilogarithm identities studied in [Kel11, KN11], we only give the minimal description here. We ask the reader to consult [KN11, Section 3] for more details.

Consider a sequence of (\mathbf{d}, \mathbf{z}) -mutations of quantum Y -seeds,

$$(4.1) \quad (B(1), Y(1)) \xleftrightarrow{\mu_{k_1}} (B(2), Y(2)) \xleftrightarrow{\mu_{k_2}} \cdots \xleftrightarrow{\mu_{k_L}} (B(L+1), Y(L+1)),$$

and suppose that it has the periodicity

$$(4.2) \quad b_{\sigma(i)\sigma(j)}(L+1) = b_{ij}(1), \quad Y_{\sigma(i)}(L+1) = Y_i(1)$$

for some permutation σ of $1, \dots, n$. Then, we have the associated sequence of (\mathbf{d}, \mathbf{z}) -mutations of (nonquantum) Y -seeds of a (nonquantum) generalized cluster algebra,

$$(4.3) \quad (B(1), y(1)) \xleftrightarrow{\mu_{k_1}} (B(2), y(2)) \xleftrightarrow{\mu_{k_2}} \cdots \xleftrightarrow{\mu_{k_L}} (B(L+1), y(L+1)),$$

and it has the same periodicity

$$(4.4) \quad y_{\sigma(i)}(L+1) = y_i(1).$$

Let us further assume the *sign-coherence property* of the sequence (4.3) (see, e.g., [Nak14]). Let ε_t and c_t ($t = 1, \dots, L$) be the *tropical sign* and the *c-vector* of $y_{k_t}(t)$ defined in [Nak14]. Let us denote the initial seed $(B(1), Y(1))$ as (B, Y) . Let $\mathbb{T}(B)$ be the *quantum torus* generated by noncommutative variables Y^α ($\alpha \in \mathbb{Z}^n$) with the relations

$$(4.5) \quad q^{\langle \alpha, \beta \rangle} Y^\alpha Y^\beta = Y^{\alpha + \beta}, \quad \langle \alpha, \beta \rangle = \sum_{i,j=1}^n \alpha_i d_i b_{ij} \beta_j.$$

We identify $Y_i = Y^{e_i}$, where e_i is the i th unit vector.

Theorem 4.1. *Under the assumption of the periodicity (4.2) and the sign-coherence property of the sequence (4.3), we have the following identities of the quantum dilogarithms of higher degrees associated to the sequence (4.1).*

(i) *Quantum dilogarithm identities in tropical form (cf. [KN11, Theorem 3.5]).*

$$(4.6) \quad \Psi_{d_{k_1}, \mathbf{z}_{k_1}, q_{k_1}}(Y^{\varepsilon_1 c_1})^{\varepsilon_1} \cdots \Psi_{d_{k_L}, \mathbf{z}_{k_L}, q_{k_L}}(Y^{\varepsilon_L c_L})^{\varepsilon_L} = 1,$$

where $Y^{\varepsilon_t c_t} \in \mathbb{T}(B)$.

(ii) *Quantum dilogarithm identities in universal form (cf. [KN11, Corollary 3.7]).*

$$(4.7) \quad \Psi_{d_{k_L}, \mathbf{z}_{k_L}, q_{k_L}}(Y_{k_L}(L))^{\varepsilon_L} \cdots \Psi_{d_{k_1}, \mathbf{z}_{k_1}, q_{k_1}}(Y_{k_1}(1))^{\varepsilon_1} = 1.$$

We omit the proof, since it is completely parallel to the one for Theorem 3.5 and Corollary 3.7 of [KN11].

Example 4.2. Let us consider the simplest nontrivial example of a generalized cluster algebra with

$$(4.8) \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{d} = (2, 1), \quad \mathbf{z} = (z_{1,1}).$$

This example was studied in [Nak14, Section 2.3] for the nonquantum case. Now let us choose

$$(4.9) \quad \mathbf{r} = (1, 2).$$

Thus, we have $q_1 = q_2 = q^2$, and the commutation relation for $Y = (Y_1, Y_2)$ is given by

$$(4.10) \quad Y_1 Y_2 = q^4 Y_2 Y_1.$$

Let us set $(B(1), Y(1)) := (B, Y)$ and consider the following sequence of mutations

$$(4.11) \quad \begin{aligned} (B(1), Y(1)) &\xrightarrow{\mu_1} (B(2), Y(2)) \xrightarrow{\mu_2} (B(3), Y(3)) \xrightarrow{\mu_1} (B(4), Y(4)) \\ &\xrightarrow{\mu_2} (B(5), Y(5)) \xrightarrow{\mu_1} (B(6), Y(6)) \xrightarrow{\mu_2} (B(7), Y(7)). \end{aligned}$$

Then, we have

$$(4.12) \quad B(t) = (-1)^{t+1} B,$$

and the quantum Y -variables mutate as follows, where we set $z = z_{1,1}$ for simplicity. (If we set $q = 1$, we recover the result in [Nak14, Table 1] for the nonquantum case.)

$$(4.13) \quad \begin{cases} Y_1(1) = Y_1 \\ Y_2(1) = Y_2, \end{cases}$$

$$(4.14) \quad \begin{cases} Y_1(2) = Y_1^{-1} \\ Y_2(2) = Y_2(1 + zq^2 Y_1 + q^4 Y_1^2), \end{cases}$$

$$(4.15) \quad \begin{cases} Y_1(3) = Y_1^{-1}(1 + q^2 Y_2 + zY_1 Y_2 + q^{-2} Y_1^2 Y_2) \\ Y_2(3) = Y_2^{-1}(1 + zq^{-2} Y_1 + q^{-4} Y_1^2)^{-1}, \end{cases}$$

$$(4.16) \quad \begin{cases} Y_1(4) = Y_1(1 + q^{-2} Y_2 + zq^{-4} Y_1 Y_2 + q^{-6} Y_1^2 Y_2)^{-1} \\ Y_2(4) = q^{-4} Y_1^{-2} Y_2^{-1}(1 + q^2 Y_2 + q^6 Y_2 + q^8 Y_2^2 \\ \quad + zY_1 Y_2 + zq^2 Y_1 Y_2^2 + q^{-4} Y_1^2 Y_2^2), \end{cases}$$

$$(4.17) \quad \begin{cases} Y_1(5) = q^{-2} Y_1^{-1} Y_2^{-1}(1 + q^2 Y_2) \\ Y_2(5) = q^{-4} Y_1^2 Y_2(1 + q^{-6} Y_2 + q^{-2} Y_2 + q^{-8} Y_2^2 \\ \quad + zq^{-4} Y_1 Y_2 + zq^{-10} Y_1 Y_2^2 + q^{-10} Y_1^2 Y_2^2)^{-1}, \end{cases}$$

$$(4.18) \quad \begin{cases} Y_1(6) = q^{-2} Y_1 Y_2(1 + q^{-2} Y_2) \\ Y_2(6) = Y_2^{-1}, \end{cases}$$

$$(4.19) \quad \begin{cases} Y_1(7) = Y_1 \\ Y_2(7) = Y_2. \end{cases}$$

Among them, the calculation of $Y_2(4)$ is the most tedious one. Now we observe the periodicity of the sequence (4.11) with $\sigma = \text{id}$. We also have the following data of the tropical signs and the c -vectors in [Nak14, Section 3.4]

$$(4.20) \quad \varepsilon_1 = \varepsilon_2 = +, \quad \varepsilon_3 = \varepsilon_4 = \varepsilon_5 = \varepsilon_6 = -,$$

$$(4.21) \quad \begin{aligned} c_1 &= (1, 0), \quad c_2 = (0, 1), \quad c_3 = (-1, 0), \\ c_4 &= (-2, -1), \quad c_5 = (-1, -1), \quad c_6 = (0, -1), \end{aligned}$$

which can be also read off from the above result by setting $q = 1$. Now, by substituting these data, the identity (4.6) reads

$$(4.22) \quad \begin{aligned} & \Psi_{2,(z),q^2}(Y_1)\Psi_{q^2}(Y_2)\Psi_{2,(z),q^2}(Y_1)^{-1} \\ & \times \Psi_{q^2}(q^{-4}Y_1^2Y_2)^{-1}\Psi_{2,(z),q^2}(q^{-2}Y_1Y_2)^{-1}\Psi_{q^2}(Y_2)^{-1} = 1, \end{aligned}$$

while the identity (4.7) reads

$$(4.23) \quad \begin{aligned} & \Psi_{q^2}(Y_2)^{-1}\Psi_{2,(z),q^2}((1+q^2Y_2)^{-1}q^2Y_2Y_1)^{-1} \\ & \times \Psi_{q^2}((1+q^2Y_2+q^6Y_2+q^8Y_2^2+zY_1Y_2+zq^2Y_1Y_2^2+q^{-4}Y_1^2Y_2^2)^{-1}q^4Y_2Y_1^2)^{-1} \\ & \times \Psi_{2,(z),q^2}((1+q^2Y_2+zY_1Y_2+q^{-2}Y_1^2Y_2)^{-1}Y_1)^{-1} \\ & \times \Psi_{q^2}(Y_2(1+zq^2Y_1+q^4Y_1^2))\Psi_{2,(z),q^2}(Y_1) = 1. \end{aligned}$$

In general, we conjecture that if the underlying nonquantum sequence (4.3) has a periodicity (4.4), then the corresponding quantum sequence (4.1) also has the same periodicity (4.2). (The converse is trivial as already stated.) This was proved for the ordinary cluster algebras in [KN11, Proposition 3.4] when $B = B(1)$ is skew-symmetric.

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